Basic Limit Ideas – 2.1, 2.2, 2.3 – the answers – and some thoughts

1) Evaluate $\lim\limits_{x\to\infty}\frac{x^2-4}{x^2+4}$. Trying to substitute large numbers, we conclude that the limit of the top approaches infinity, as does the limit of the bottom. This $\frac{\infty}{\infty}$ idea **doesn't tell us anything** – it hides what is really happening. So we need to do algebra to figure it out. Of course, we always have the PreCalc rule to fall back on – and the coefficients of the (identical) high degree terms of top and bottom tell us that the limit is 1. The algebra way: $\lim\limits_{x\to\infty}\frac{x^2-4}{x^2+4}=\lim\limits_{x\to\infty}\frac{x^2\left(1-1/x^2\right)}{x^2\left(1+1/x^2\right)}=\lim\limits_{x\to\infty}\frac{\left(1-1/x^2\right)}{\left(1+1/x^2\right)}=1.$

2) Evaluate $\lim\limits_{x\to3+}\frac{x-3}{x^2-2x-3}$. Substituting 3 for *x* gives us $\frac{0}{0}$ - so we don't know anything. But the very fact that we GET **0 over 0** tells us we have **common factors** we can cancel.

$$\lim_{x\to3+}\frac{x-3}{x^2-2x-3}=\lim_{x\to3+}\frac{x-3}{(x-3)(x+1)}=\lim_{x\to3+}\frac{1}{x+1}=\frac{1}{4}$$

3) Evaluate $\lim\limits_{x\to-4}\frac{|x+4|}{x+4}$. When we substitute we get the nearly ubiquitous $\frac{0}{0}$ - so we have to try something else. Here we need to – somehow – consider the idea or definition of absolute value. For the *x* value of -4, this corresponds exactly to the idea of 'consider both one sided limits'. So – **the left hand limit:** $\lim\limits_{x\to-4^-}\frac{|x+4|}{x+4}=\lim\limits_{x\to-4^-}\frac{-(x+4)}{x+4}=\lim\limits_{x\to-4^-}(-1)=-1$. This is because when we consider *x*'s smaller than -4, we are taking the absolute value of a negative number, and to make that negative number positive I can just take the opposite of it. Now let's consider **the right hand limit:** $\lim\limits_{x\to-4^+}\frac{|x+4|}{x+4}=\lim\limits_{x\to-4^+}\frac{(x+4)}{x+4}=\lim\limits_{x\to-4^+}1=1$. What are we being asked in this question? We are being asked if the **two-sided** limit exists. A two-sided limit exists only when both one-sided limits give the same result. Here, -1 is NOT the same as +1. So this limit is **dne** – does not exist.

4) Evaluate $\lim\limits_{x\to-3^-}\frac{\sqrt{x^2-9}}{x-3}$. Substitution yields … the answer!! $\lim\limits_{x\to-3^-}\frac{\sqrt{x^2-9}}{x-3}=\frac{\sqrt{(-3)^2-9}}{-3-3}=\frac{0}{-6}=0$

5) Evaluate $\lim\limits_{x\to7}\frac{\sqrt{7}-\sqrt{x}}{7-x}$. Here substitution doesn't tell us anything. So we need an idea to cancel out the 0 over 0. You have already been taught such an idea: multiplying by the conjugate! So we have

$$\lim_{x\to7}\frac{\left(\sqrt{7}-\sqrt{x}\right)}{(7-x)}\bullet\frac{\left(\sqrt{7}+\sqrt{x}\right)}{\left(\sqrt{7}+\sqrt{x}\right)}=\lim_{x\to7}\frac{(7-x)}{(7-x)\left(\sqrt{7}+\sqrt{x}\right)}=\lim_{x\to7}\frac{1}{\left(\sqrt{7}+\sqrt{x}\right)}=\frac{1}{2\sqrt{7}}.$$

6) Evaluate $\lim\limits_{x\to-\infty}\frac{1}{x}$. = 0. This standard 'limit to infinity' you probably learned already in PreCalculus. As you divide by an increasingly larger number the whole ratio approaches 0. If I invited you to my hovel for pizza, and somehow cut the pizza up into 1,000,000 pieces, would you want seconds, or would you decide not even to waste your time?

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7) Evaluate $\lim_{x\to\infty}\sqrt{x^2+x}-x$. This is a difficult one. Even for those of you who **think** of the multiplying by conjugate strategy, you aren't used to applying it when there is no fraction. But that's what you do here. The idea is – instead of the infinity – infinity deal, you want to **turn this into a fraction** so that you have one thing that's a top over a bottom. Let's see how to do this:

$\lim_{x\to\infty}\frac{\left(\sqrt{x^2+x}-x\right)}{1}\bullet\frac{\left(\sqrt{x^2+x}+x\right)}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{(x^2+x)-(x^2)}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}$. Now what? Now we use the other strategy we know when a radical is involved: Divide top and bottom by $|x|$. Alternatively, you could factor an *x*-like thing out of the top and bottom ….

$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}\bullet\frac{1/|x|}{1/|x|}=\lim_{x\to\infty}\frac{1}{\sqrt{\frac{x^2+x}{x^2}}+1}=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1}=\frac{1}{2}$. You'll notice we change the *representation* of $|x|$ - in fact we use two different and equivalent representations. That allows the simplification ….

8) Evaluate $\lim_{x\to-\infty}\frac{2-y}{\sqrt{7+6y^2}}$. Here's an example where we use the strategy of dividing by absolute of y … but in a situation where *y* is always negative.

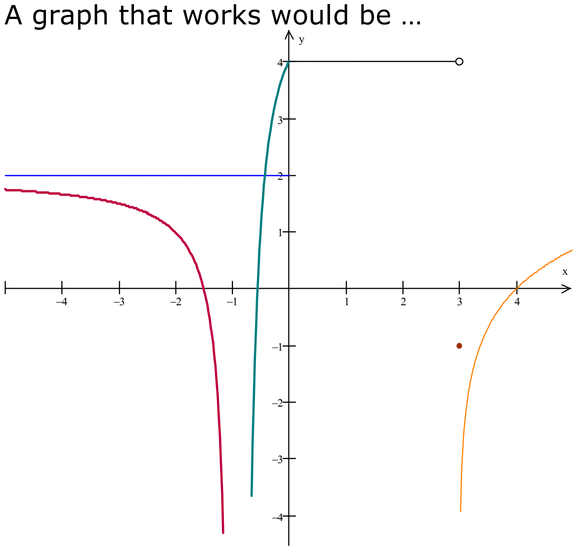
$$\lim_{x\to-\infty}\frac{2-y}{\sqrt{7+6y^2}}\bullet\frac{1/|y|}{1/|y|}=\lim_{x\to-\infty}\frac{2-y}{\sqrt{7+6y^2}}\bullet\frac{1/(-y)}{1/\sqrt{y^2}}=\lim_{x\to-\infty}\frac{\frac{2}{-y}+\frac{y}{y}}{\sqrt{\frac{7}{y^2}+6\frac{y^2}{y^2}}}=\lim_{x\to-\infty}\frac{\frac{2}{-y}+1}{\sqrt{\frac{7}{y^2}+6}}=\frac{1}{\sqrt{6}}$$

9) Evaluate $\lim_{x\to3}\frac{x^2-2x}{x+1}$. $\lim_{x\to3}\frac{x^2-2x}{x+1}=\frac{(3)^2-2(3)}{3+1}=\frac{3}{4}$.

10) Graph *one* function that matches all of the following criteria:

* $f(0)=4, f(3)=-1$
* $\lim_{x\to-1}f(x)=-\infty$
* $\lim_{x\to-\infty}f(x)=2, \lim_{x\to\infty}f(x)=+\infty$
* $\lim_{x\to3-}f(x)=4; \lim_{x\to a3+}f(x)=-\infty$

A graph that works would be …



Note that $\lim_{x\to-1}f(x)=-\infty$ means that **both** one-sided limits approach negative infinity.

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11) Let $f(x)=\begin{cases}\dfrac{x^2-x}{2x}, & x>0\\ k, & x\le 0\end{cases}$. Select a value of k so that $\lim\limits_{x\to 0} f(x)$ exists. AND tell me what the limit is.

You are asked to find the two-sided limit of *f*. The two-sided limit exists only when both one-sided limits exist. $\lim\limits_{x\to 0^-} k=k$. $\lim\limits_{x\to 0+}\dfrac{x^2-x}{2x}=\lim\limits_{x\to 0+}\dfrac{x(x-1)}{x(2)}=\lim\limits_{x\to 0+}\dfrac{(x-1)}{(2)}=\dfrac{-1}{2}$. If the two-sided limit is to exist, then we must have $k=\dfrac{-1}{2}$; If we do that, then the limit is obviously $\dfrac{-1}{2}$.

Some thoughts:

There were several papers where the student looked quickly at a problem and then wrote down an answer. Now – they may have had no idea and that was the best they could do. I don't really know – it LOOKS like working quickly to me. Be careful of that …

Try SUBSTITUTION first!

I still have folks looking at $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ and just writing 0 or ∞ or **dne**. Please stop – those 'indeterminate forms' (along with $\infty-\infty$) just tell you that another way to look at the problem must be found.

There also seems to be a habit (my perception) of trusting your intuition a little too much – you will try a strategy and, assuming that's the correct one, will interpret whatever you get as the correct answer. One of the things that distinguishes great mathematicians is **knowing** whether they've found the solution.

You may **NOT** just square the top and bottom. That was a great strategy when you were solving equations and could square both sides, but you can't just square a fraction and expect it to remain the same number! ! And it's not even a strategy that works …. Consider $f(x)=\begin{cases}+1, & \text{when } x \text{ is rational}\\ -1, & \text{when } x \text{ is irrational}\end{cases}$.

Then *f* does not have a limit as x -> 3; **but** *f*^2 does have a limit as x -> 3.